数理解析研究所講究録
第 738 巻 1991 年 1-9

A modulus of uniform continuity with some order in $L^s_{loc}(\Omega\,;R^N)$ $(2\leq s<\infty)$ and a sharp estimate of *Lebesgue points* of the first-derivatives of minimizers of a Quasi-convex functional in the calculus of variations .

堀畑 和弘

KAZUHIRO HORIHATA

Department of Mathematics, Faculty of Science and Technology ,
Keio University

Abstract. This paper establishes that minimizers of *strictly quasi-convex* variational functionals , satisfy a modulus of uniform continuity with some order in the norm of $L^s_{loc}(\Omega\,;R^N)$ with $2\leq s\leq\infty$. This modulus of uniform continuity combined with a result in the present author's paper and one of Evans's results implies a local Hölder continuity and a sharp estimate for the *Hausdorff dimension* of *Lebesgue points* of the first derivatives of minimizers .

1. INTRODUCTION

In this paper we establish that minimizers for certain functionals in the calculus of variations satisfy a modulus of uniform continuity of some order in the norm of $L^s_{loc}(\Omega\,;R^N)$ with $2\leq s<+\infty$. This functional is given as follows : Let n,N be positive integers . We denote by $M^{n\times N}$ the space of all real $n\times N$ matrices and suppose that $\Omega\subset R^n$ is a bounded with smooth boundary . Then for $v:\Omega\mapsto R^N$, we consider the functional

$$I[v]\equiv\int_\Omega F(\nabla v)dx\,, \tag{1.1}$$

where $v=(v^i)$, $\nabla v=(\partial v^i/\partial x_\alpha)$ $(\alpha=1,\cdots,n\,,\,i=1,\cdots,N)$ is the gradient matrix of v and $F:M^{n\times N}\mapsto R$ is any given mapping , which is strictly defined later. Here we introduce another notation which will be used in this paper : $L^s(\Omega\,;R^N)$ is sth$-$ power integrable function space. We also denote by $L^s_{loc}(\Omega\,;R^N)$ locally sth$-$ power *integrable function space* . $H^{1,s}(\Omega\,;R^N)$ *and* $\overset{\circ}{H}{}^{1,s}(\Omega\,;R^N)$ *are the usual Sobolev spaces.* Also $|A|$ and $H^\gamma(A)$ means *the Lebesgue measure* and *the* $\gamma-$ *dimensional Hausdorff measure* of measurable set A in R^n , respectively , (see *Giaquinta* [Gm1] and *Giusti* [Gi] for detailed definition).

We introduce a forward translation operator and also a forward difference operator of a map in $f\in L^s(\Omega\,;R^N)$: Let h be any small number and e be a unit vector in R^n . We define a forward translate operator $\cdot^+$ by

$$f^+(x)\equiv f(x+he) \tag{1.2}$$

and define a forward difference operator τ_h by

$$\tau_h f=f^+-f\,. \tag{1.3}$$

We adopt the summation convention : For $^\forall A, P, Q \in M^{n\times N}$, we define

$$DF(A) = (\frac{\partial F}{\partial p^i_\alpha}(A)),$$

$$D^2F(A) = (\frac{\partial^2 F}{\partial p^i_\alpha \partial^j_\beta}(A))$$

$$(\alpha, \beta = 1, \cdots, n\,, i, j = 1, \cdots, N),$$

$$DF(A)\cdot P = \sum_{\alpha=1}^{n}\sum_{i=1}^{N}\frac{\partial F}{\partial p^i_\alpha}(A)P^i_\alpha\,,$$

and

$$D^2F(A) < P, Q >= \sum_{\alpha,\beta=1}^{n}\sum_{i,j=1}^{N}\frac{\partial^2 F}{\partial p^i_\alpha \partial p^j_\beta}(A)P^i_\alpha Q^j_\beta\,.$$

Let $F(x,z,p) : \Omega \times R^N \times M^{n\times N} \mapsto R$ be a function satisfying

(H1) $F(x,z,p) \leq K[1+p^s]$

(H2) $F(x,z,p) \geq m$

(H3) $|F(x,z,p_1) - F(x,z,p_2)| \leq K[1 + |p_1|^{s-1} + |p_2|^{s-2}]|p_1 - p_2|$

(H4) $|F(x_1,z_2,p) - F(x_1,z_2,p)| \leq K[1+|p|^s][|x_1 - x_2| + |z_1 - z_2|]$

for $^\exists m\, K > 0$ and $s(1 \leq s < \infty)$. The first question in the calculus of variations can be considered as the existence problem of minimizers in some function space . Under the above condition , *Morrey* [Mo] has isolated that a necessary and sufficient condition of certain functional $F(x,z,p)$ for the lower semicontinuity of $I[\cdot]$ on some *Sobolev space* is *quasi-convex* :

$$\int_O F(x_0,z_0,p_0)dy \leq \int_O F(x_0,z_0,p_0 + \nabla\phi)dy \quad \text{for } ^\forall(x_0,z_0,p_0) \in \Omega \times R^N \times M^{n\times N},$$

for an arbitrary smooth , bounded , open set $O \subset R^n$, $^\forall A \in M^{n\times N}$ and $^\forall \phi \in C^1_0(O\,;R^N)$.

Recently *Acerbi* and *Fusco* [AF] has refined *Morrey's theorem* , who have obtained the following for $F(p)$:

THEOREM 0 ([AF]). *Assume that $F : M^{n\times N} \mapsto R$ is continuous and for some positive numbers C and s the following*

$$0 \leq F(p) \leq C(1+|p|^s)$$

holds for $^\forall p \in M^{n\times N}$. Then $I[\cdot]$ is weakly sequentially lower semicontinuous on the Sobolev space $H^{1,m}(\Omega\,;R^N)$ if and only if F is quasi-convex.

Also the second question can be considered as the regularity problem of such minimizers . However one often encounters that a minimizer is not necessarily regular everywhere in Ω , even when F is uniform convex (see [Gm1] , [Gm2] , [Gm3] , [GG2] and [GI]). For the study of partial regularity , *Evans* [Ev] (see also [EG] and [GM]) has showed that minimizers has Hölder continuous first derivatives on some open subset $\Omega_0 \subset \Omega$ satisfying $|\Omega/\Omega_0| = 0$, when $F \in C^2(M^{n\times N}\,; R^N)$ and $D^2F(p)$ is uniform continuous in $M^{n\times N}$ and strictly *quasi - convex* : For $^\exists\gamma > 0$ and $^\exists s$ $(2 \leq s < \infty)$ F satisfies

$$\gamma\int_\Omega (1+|\nabla\phi|^{s-2})|\nabla\phi|^2 dy \leq \int_\Omega [F(A+\nabla\phi) - F(A)]dy \tag{1.4}$$

$$\text{for } {}^\forall A \in M^{n\times N} \text{ and } {}^\forall \phi \in \overset{\circ}{C}{}^1(\Omega\,;\,R^N)\,.$$

and suppose that

$$\text{(H5)} \qquad |D^2F(p)| \,\le\, C_0(1+|p|^{s-2})$$

for some constant C_0 and ${}^\forall p \in M^{n\times N}$.

We remark that assumption $(H5)$ implies that there exist positive constants C_1 and C_2 such that

$$\text{(H6)} \qquad |F(p)| \,\le\, C_1(1+|p|^s)$$

$$\text{(H7)} \qquad |DF(p)| \,\le\, C_2(1+|p|^{s-1})$$

for all $p \in M^{n\times N}$. Under the above condition , *Evans* has proved

THEOREM 1 ([Ev]). *Assume that* $2 \le s < +\infty$, *the function* F *satisfies (1.5) and* $(H5)$. *Let* $u \in H^{1,s}(\Omega\,;R^N)$ *be a minimizer of* $I[\cdot]$. *Then there exists an open subset* Ω_0 *of* Ω *such that*

$$\text{(1.5)} \qquad |(\Omega/\Omega_0)| \,=\, 0$$

and the first derivatives of a minimizer u *are locally Hölder continuous on* Ω_0 :

$$\nabla u \in C^\alpha(\Omega_0\,;\,M^{n\times N})$$

for each $0 < \alpha < 1$.

This proof is performed by combining *a blow-up argument* with *the following Caccioppoli inequality* :

THEOREM 2 ([Ev]). *There exists a constant* C_3 *independent of* r *such that a minimizer* u *satisfies*

$$\text{(1.6)} \quad \int_{B_{r/2}(x)} (1+|\nabla u|^{s-2})|\nabla u|^2 dx \,\le\, C_3[(1/r)^2 \int_{B_r(x)} |u-a|^2 dx \,+\, (1/r)^s \int_{B_r(x)} |u-a|^s dx]$$

$$\text{for} \qquad {}^\forall B_r(x) \subset\subset \Omega \text{ and } {}^\forall a \in R^N\,.$$

From Theorem 2 and *a Gehring inequality* [Gm] , it follows that

THEOREM 3. *When* ∇u *satisfies the inequality (1.7) of Theorem 2 , there exist positive numbers* t $(t > s)$*depending only on* C_3, s, Ω *and* C_4 *depending only on* $C_3, s,$ Ω *and* $\tilde\Omega$ *such that* $\nabla u \in L^t_{loc}(\Omega\,;R^N)$ *and moreover the following holds :*

$$\text{(1.7)} \qquad [\frac{1}{|\tilde\Omega|}\int_{\tilde\Omega}(1+|\nabla u|)^t dx]^{1/t} \,\le\, C_4[\frac{1}{|\Omega|}\int_{\Omega}(1+|\nabla u|)^s dx]^{1/s}$$

$$\textit{for} \qquad {}^\forall\tilde\Omega \subset\subset \Omega\,.$$

2. MAIN RESULT

Now we can state the main theorem

THEOREM 4 (MAIN THEOREM). *Assume that* $2 \le s < +\infty$, *the function* F *satisfies (1.4) and (H5). Let* u *be minimizer of* $I[\cdot]$ *in* $H^{1,s}(\Omega\,;\,R^N)$. *Then for an arbitrary open set* $\tilde{\Omega}$ *compactly contained in* Ω , *the following holds :*

$$\int_{\tilde{\Omega}} |\tau_h \nabla u|^2 dx \le C_5 \cdot h \qquad \text{for} \quad 0 <^{\forall} h < \frac{1}{8} dist(\tilde{\Omega}, \partial\Omega)\,, \tag{2.1}$$

where C_5 *is a constant depending only on* n , N , γ , C_0 , $||\nabla u||_{L^s}$, $\tilde{\Omega}$ *and* Ω .

Here we notice that as in the same way of author's previous result , one finds

THEOREM 5([Ho]). *Let* f *be a function belonging to* $L^p_{loc}(\Omega\,;R^N)$ $(1 \le p < \infty)$ *with the following condition: Let* $\tilde{\Omega}$ *be an arbitrary open set compactly contained in* Ω *and suppose that there exist positive numbers* C_6 *and* α $(0 < \alpha < n/p)$ *independent of* h *such that* f *satisfies*

$$\int_{\tilde{\Omega}} |\, \tau_h f \,|^p \, dx \le C_6 \cdot h^{p\alpha} \tag{2.2}$$

for any number h *with* $0 < h < \frac{1}{4}$ $dist(\tilde{\Omega}, \partial\Omega)$. *Then for the singular set* S_f *of the map* f *defined by*

(2.3)

$$S_f = \{x \in \Omega : \nexists \lim_{\rho \to +0} f_{x,\rho}\} \cup \{x \in \Omega : \lim_{\rho \to +0} |f_{x,\rho}| = +\infty\} \cup \{x \in \Omega : \lim_{\rho \to +0} \int_{B_\rho(x)} |f - f_{x,\rho}|^p dy > 0\}$$

where $f_{x,\rho} = 1/|B_\rho| \int_{B_\rho(x)} f(y)dy$, *the following holds:*

$$H^{(\beta)}(S) = 0 \tag{2.4}$$

for any positive number β *with* $n - p\alpha < \beta$.

From Theorem 4 and Theorem 5 , we obtain

THEOREM 6. *A singular set* $S_{\nabla u}$ *of the first derivatives of such minimizers , have at most*

$$H^{n-1+\epsilon}(S) = 0 \tag{2.5}$$

$$\text{for} \quad {}^{\forall}\epsilon > 0\,.$$

In addition , noting [Ev] and [EG] , one finds that (2.5) shows the first derivatives of minimizers satisfy local Hölder continuity on Ω/S :

$$\nabla u \in C^{\alpha}(\Omega/S\,; M^{n \times N}) \qquad \text{for} \qquad 0 <^{\forall} \alpha < 1\,.$$

3. PROOF OF THEOREM 4

Since u is a minimizer of $I[\cdot]$ in $H^{1,s}(\Omega\,;\,R^N)$, u satisfies the following first-variational formula :

$$\int_{\Omega} DF(\nabla u) \cdot \nabla \phi dx = 0 \qquad \text{for} \qquad {}^{\forall}\phi \in \overset{\circ}{H}{}^{1,s}(\Omega\,;R^N)\,. \tag{3.1}$$

Transferring x to $x+he$ along the direction of a unit vector e, we have

$$\int_\Omega DF(\nabla u^+)\cdot\nabla\phi dx = 0 \qquad \text{for} \qquad {}^\forall\phi\in \overset{\circ}{H}{}^{1,s}(\Omega_1\,;R^N). \tag{3.2}$$

where $\Omega_0 = \tilde{\Omega}$, $\Omega_k = \{x\in\Omega : dist(x,\tilde{\Omega}) < \frac{k}{4}dist(\tilde{\Omega},\partial\Omega)\}$ $(k=0,1,\cdots,4)$. (3.1) subtracted after (3.2) gives

$$\int_\Omega [DF(\nabla u^+) - DF(\nabla u)]\cdot\nabla\phi dx = 0 \qquad \text{for} \qquad {}^\forall\phi\in \overset{\circ}{H}{}^{1,s}(\Omega_1\,;R^N). \tag{3.3}$$

Thus we have

$$\int_\Omega\int_0^1 D^2F(\nabla u + t\nabla(\tau_h u)) < \nabla(\tau_h u),\nabla\phi > dtdx = 0 \qquad \text{for} \qquad {}^\forall\phi\in \overset{\circ}{H}{}^{1,m}(\Omega_1\,;R^N). \tag{3.4}$$

Substituting $\tau_h u\eta^2$ for ϕ, where a cut-off function $\eta\in C_0^\infty(\Omega)$ satisfies

$$\eta = \begin{cases} 1 & \text{in } \ \Omega_0\,, \\ 0 & \text{outside } \ \Omega_1 \end{cases} \qquad \text{with} \qquad \begin{cases} |\nabla\eta| \le \frac{2}{dist(\Omega_0,\Omega_1)}\,, \\ 0\le|\eta|\le 1\,. \end{cases}$$

We can proceed the calculation of (3.3) as follows :

$$\begin{aligned} &\int_\Omega < \tau_h[DF(\nabla u)], \nabla(\tau_h u)\eta^2) > dx \\ &= \int_\Omega\int_0^1 D^2F(\nabla u + t\nabla(\tau_h u)) \\ &[< \nabla(\tau_h u),\nabla(\tau_h u)\eta^2 > +2 < \nabla(\tau_h u),\tau_h u\eta\nabla\eta >]dtdx \end{aligned} \tag{3.5}$$

Consequently , the following

$$\begin{aligned} &\int_\Omega D^2F(A) < \nabla(\tau_h u),\nabla(\tau_h u) > \eta^2 dx \\ &= \int_\Omega [D^2F(A) - \int_0^1 D^2F(\nabla u + t\nabla(\tau_h u))] \\ &< \nabla(\tau_h u),\nabla(\tau_h u) > \eta^2 dtdx \\ &- 2\int_\Omega\int_0^1 D^2F(\nabla u + t\nabla(\tau_h u))dt < \nabla(\tau_h u)\eta, \tau_h u\nabla\eta > dtdx\,, \end{aligned} \tag{3.6}$$

holds for ${}^\forall A\in M^{n\times N}$. Now let Ω_1 be approximated by a union of hypercubes $D_{k,i}$ with each edge length $1/k$ sufficiently large $k>0$:

$$\begin{aligned} &\Omega_1 \subset \bigcup_{i=1}^{I} D_{k,i} \qquad \text{with} \qquad \Omega_1\subset H_k\subset\Omega_2\,, \\ &\overset{\circ}{D}_{k,i}\cap\overset{\circ}{D}_{k,j} = \varnothing \quad \text{in} \quad i\neq j\,, \\ &|H_k - \Omega_2| \to 0 \quad \text{as} \quad k\to+\infty\,, \\ &|D_{k,i}| = (1/k)^n\,. \end{aligned} \tag{3.7}$$

Moreover we remark that there exists subsequence of I which we call $I(k)$ such that $H_k = \bigcup_{i=1}^{I(k)} D_{k,i}$ satisfies $\Omega_1 \subset H_k \subset \Omega_2$ and $|\Omega_2 - H_k| \to 0$ as $k \to +\infty$. For $x \in H_k$, we define

$$\overline{\nabla u}(x) \equiv \frac{1}{|D_{k,i}|}\int_{D_{k,i}} \nabla u(y)dy \quad \text{for} \quad x \in D_{k,i} \quad \text{and}\, i = 1, \cdots, I.$$

When we adopt $\overline{\nabla u}(x) + s\overline{\nabla(\tau_h u)}(x) \quad (0 \le s \le 1)$, $\overline{\nabla \tau_h u}(x) \equiv \overline{\nabla u^+}(x) - \overline{\nabla u}(x)$ as A , then it follows from (3.5) , (3.6) and (3.7) that

$$\begin{aligned}
&\int_\Omega D^2F(\overline{\nabla u} + s\overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dx \\
&= \int_\Omega D^2F(\overline{\nabla u} + s\overline{\nabla(\tau_h u)})dx - \int_0^1\int_\Omega D^2F(\nabla u + t\nabla(\tau_h u)) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dtdx \\
&\quad - 2\int_\Omega\int_0^1 D^2F(\nabla u + t\nabla(\tau_h u)) < \nabla(\tau_h u)\eta, \tau_h u \nabla\eta > dtdx .
\end{aligned} \tag{3.8}$$

By integrating (3.8) over $[0,1]$ for s , we obtain

$$\begin{aligned}
&\int_\Omega\int_0^1 D^2F(\overline{\nabla u} + s\overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dsdx \\
&= \int_\Omega\int_0^1 [D^2F(\overline{\nabla u} + t\overline{\nabla(\tau_h u)}) - D^2F(\nabla u + t\nabla(\tau_h u))] < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dtdx \\
&\quad - 2\int_\Omega\int_0^1 (D^2F)(\nabla u + t\nabla(\tau_h u))dt < \nabla(\tau_h u)\eta, (\tau_h u)\nabla\eta > dx .
\end{aligned} \tag{3.9}$$

The above (3.9) is a starting point to our proof. The original technique used here is seen in [Da] and [Mo] . At first , we estimate the left-hand side in (3.9) from below :

$$\begin{aligned}
&\int_\Omega\int_0^1 D^2F(\overline{\nabla u} + s\overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dsdx \\
&\ge \int_0^1\int_{H_k} D^2F(\overline{\nabla u} + s\overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dsdx \\
&\quad + \int_0^1\int_{\Omega_2/H_k} D^2F(\overline{\nabla u} + s\overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dsdx \\
&= \sum_{i=1}^{I(k)}\int_0^1\int_{D_{k,i}} D^2F(\overline{\nabla u} + s\overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dsdx \\
&\quad + \int_0^1\int_{\Omega_2/H_k} D^2F(\overline{\nabla u} + s\overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dsdx .
\end{aligned} \tag{3.10}$$

If we use the mean value theorem for s , then there exist positive numbers $s_{0,i}$ $(i = 1, \cdots, I(k))$ such that

$$\begin{aligned}
&= \sum_{i=1}^{I(k)}\int_{D_{k,i}} D^2F(\overline{\nabla u} + s_{0,i}\overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dsdx \\
&\quad + \int_0^1\int_{\Omega_2/H_k} D^2F(\overline{\nabla u} + s\overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dsdx
\end{aligned}$$

Here we remark that from *Morrey* ([Mo] , Th 4.4.3) and *Federer* ([Fe] , Th 5.1.10) assumption (1.4) implies the strong *Legendre - Hadamard* condition :

$$\sum_{\alpha,\beta}\sum_{i,j}\frac{\partial^2 F}{\partial p^i_\alpha \partial p^j_\beta}(A)\xi_\alpha\xi_\beta\eta^i\eta^j \ge \gamma|\xi|^2|\eta|^2 \tag{3.11}$$

for $^\forall A \in M^{n\times N}$, $^\forall\xi \in R^n$ and $^\forall\eta \in R^N$.

Thus by noting that $\overline{\nabla u}$ is a constant on each hypercube $D_{k,i}$ $(i = 1, \cdots, I)$, we have

$$(3.10) \ge \gamma\sum_{i=1}^{I(k)}\int_{D_{i,k}}|\nabla(\tau_h u)|^2dx + \int_0^1\int_{\Omega_2/H_k} D^2F(\overline{\nabla u} + s\overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dsdx$$

(3.12)

$$= \gamma\int_{H_k}|\nabla(\tau_h u)|^2dx\,. + \int_0^1\int_{\Omega_2/H_k} D^2F(\overline{\nabla u} + s\overline{\nabla(\tau_h u)}) < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dsdx\,.$$

Next we estimate the first term on the right - hand side in (3.9) : From uniform continuity assumption of $D^2F(p)$, there exists a non-negative function $w(t)$ increasing in t , and $w(0) = 0$ concave , continuous and bounded and a constant C_7 , such that we obtain

$$\int_\Omega\int_0^1[D^2F(\overline{\nabla u} + s\overline{\nabla(\tau_h u)}) - D^2F(\nabla u + s\nabla(\tau_h u))] < \nabla(\tau_h u), \nabla(\tau_h u) > \eta^2 dtdx$$
$$\le C_7\int_{\Omega_1}\int_0^1[1 + |\overline{\nabla u} + s\overline{\nabla(\tau_h u)}|^{s-2} + |\nabla u + s\nabla(\tau_h u)|^{s-2}]$$
$$w(|\overline{\nabla u} - \nabla u|^2 + |\overline{\nabla u^+} - \nabla u^+|^2)|\nabla(\tau_h u)|^2dx$$
$$\le 2C_7 2^{s-1}\int_{\Omega_1}[1 + |\overline{\nabla u}|^{s-2} + |\overline{\nabla u^+}|^{s-2} + |\nabla u|^{s-2} + |\nabla u^+|^{s-2}]$$
$$[|\nabla u|^2 + |\nabla u^+|^2]\cdot w(|\overline{\nabla u} - \nabla u|^2 + |\overline{\nabla u^+} - \nabla u^+|^2)dx\,. \tag{3.13}$$

Since $\nabla u \in L^t_{loc}(\Omega\,; R^N)$ $(t > s)$ from (1.8) of Theorem 3 , we can apply Hölder inequality to (3.13) as follows : For $s_1 = t/(s-2)$, $s_2 = t/2$ and $s_3 = t/(t-s)$, we estimate the right-hand in (3.13)

$$\le 2^sC_7 5\cdot 2\{\int_{\Omega_1}[1 + |\overline{\nabla u}|^t + |\overline{\nabla u^+}|^t + |\nabla u|^t + |\nabla u^+|^t]dx\}^{(s-2)/t}$$
$$\{\int_{\Omega_1}[|\nabla u|^t + |\nabla u^+|^t]dx\}^{2/t}\{\int_{\Omega_1}w^{t/(t-s)}(|\overline{\nabla u} - \nabla u|^2 + |\overline{\nabla u^+} - \nabla u^+|^2)\,dx\}^{(t-s)/t}\,.$$

Successively by using bounded and concave properties of $w(t)$, we have

$$\leq 2^s 10 C_7 \{\int_{\Omega_2} [1 + |\overline{\nabla u}|^t + |\nabla u|^t] dx\}^{(s-2)/t}$$

$$\{\int_{\Omega_2} |\nabla u|^t\, dx\}^{2/t} \{\int_{\Omega_1} w(|\overline{\nabla u} - \nabla u|^2 + |\overline{\nabla u^+} - \nabla u^+|^2)\, dx\}^{(t-s)/t}$$

$$\leq 2^s 10 C_7 |\Omega_1|^{(t-s)/t} \{\int_{\Omega_2} [1 + |\overline{\nabla u}|^t + |\nabla u|^t] dx\}^{s/t}$$

$$\{\frac{1}{|\Omega_1|}\int_{\Omega_1} w(|\overline{\nabla u} - \nabla u| + |\overline{\nabla u^+} - \nabla u^+|)\, dx\}^{(t-s)/t}$$

$$\leq 2^s C_7 10 |\Omega|^{1-s/t} \{\int_{\Omega_2} [1 + |\overline{\nabla u}|^t + |\nabla u|^t] dx\}^{s/t}$$

$$\cdot\, w(\frac{1}{|\Omega_1|}\int_{\Omega_1} [|\overline{\nabla u} - \nabla u| + |\overline{\nabla u^+} - \nabla u^+|\,] dx)^{(t-s)/t}. \tag{3.14}$$

From L_1 - norm continuity of integrable function , for $^\forall \epsilon > 0$, there exists $k = k(\epsilon)$ such that

$$(3.14) \leq 2^s 10 C_7 |\Omega|^{1-s/t} \cdot \epsilon \cdot \{\int_{\Omega_2} [1 + |\overline{\nabla u}|^t + |\nabla u|^t] dx\}^{s/t}. \tag{3.15}$$

Finally we shall estimate the second term on the right-hand side in (3.9) : From assumption (H5) and using *Newton - Leibnitz* formula we obtain

$$-2\int_{\Omega_1}\int_0^1 (D^2F)(\nabla u + t\nabla(\tau_h u))dt < \nabla(\tau_h u)\eta, \tau_h u \nabla\eta > dtdx$$

$$\leq 2C_0 \int_{\Omega_1}\int_0^1 (1 + |\nabla u + t\nabla(\tau_h u)|^{s-2})|\nabla(\tau_h u)| \cdot |\tau_h u| \cdot |\nabla \eta| dx$$

$$\leq 2^s C_0 \int_{\Omega_1} (1 + |\nabla u^+|^{s-2} + |\nabla u|^{s-2})|\nabla(\tau_h u)| \cdot |\tau_h u| \cdot |\nabla\eta| dx$$

$$\leq 2^s C_0 \frac{2}{dist(\Omega_0, \Omega_1)} \{\int_{\Omega_1} [1 + |\nabla u^+|^{s-2} + |\nabla u|^{s-2}]^{s/(s-2)} dx\}^{(s-2)/s}$$

$$\{\int_{\Omega_1} [|\nabla u^+| + |\nabla u|]^s dx\}^{1/s} \{\int_{\Omega_1} |\tau_h u|^s dx\}^{1/s}$$

$$\leq 2^s C_0 3 \cdot 2 \frac{2}{dist(\Omega_0, \Omega_1)} \{\int_{\Omega_2} [1 + |\nabla u|^s] dx\}^{1-1/s} \{\int_{\Omega_1} |\tau_h u|^s dx\}^{1/s} \tag{3.16}$$

$$\leq 2^s 12 C_0 \frac{h}{dist(\Omega_0, \Omega_1)} \{\int_{\Omega_2} [1 + |\nabla u|^s] dx\}^{1-1/s} \{\int_{\Omega_2} |\nabla u|^s dx\}^{1/s}.$$

Consequently it follows from (3.12) , (3.15) and (3.16) that

$$\gamma \int_{H_k} |\nabla(\tau_h u)|^2 dx$$

$$\leq 2^s 10 C_7 |\Omega_1|^{1-s/t}\, \epsilon \{\frac{1}{|\Omega_1|}\int_{\Omega_2} (1 + |\nabla u|^t + |\overline{\nabla u}|^t) dx\}^{s/t}$$

$$+ \frac{12 2^s C_0 h}{dist(\Omega_0, \Omega_1)} \{\int_{\Omega_2} (1 + |\nabla u|^s) dx\}^{1-1/s} \{\int_{\Omega_2} |\nabla u|^s dx\}^{1/s}. \tag{3.17}$$

Now letting pass to the limit $k \to \infty$, we deduce the desired estimates :

$$\int_{\tilde{\Omega}} |\nabla(\tau_h u)|^2 dx \leq \gamma^{-1} \frac{2^s 80 C_0 h}{dist(\tilde{\Omega}, \partial\Omega)} \{ \int_{\Omega_2} (1 + |\nabla u|^s) dx \} . \tag{3.18}$$

This completes our proof.

Acknowledgement

The author is very grateful to Prof. N.KIKUCHI for especially drawing my attention to the problem and his constant encouragement .

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